

June 27, 2014

# Nonmeasurable subgroups of compact groups

by Salvador Hernández, Karl H. Hofmann, and Sidney A. Morris

## Abstract.

In 1985 S. Saeki and K. Stromberg published the following question: *Does every infinite compact group have a subgroup which is not Haar measurable?* An affirmative answer is given for all compact groups with the exception of some metric profinite groups known as strongly complete. In this spirit it is also shown that every compact group contains a non-Borel subgroup.

## Introduction

For a compact group  $G$ , *measurable* means “measurable with respect to the unique normalized Haar measure  $\mu$  on  $G$ .” Since Haar measure is a Borel measure, every Borel subset of  $G$  is measurable. A subset  $S \subseteq G$  is a *null set* if  $\mu(K) = 0$  for each compact subset  $K$  of  $S$ , and if for each  $\epsilon > 0$  there is an open neighborhood of  $S$  such that  $\mu(U) < \epsilon$ . Every subset of a null set is measurable (see [2], Paragraph after Chap IV, §5, n° 2, Definition 3, p. 172, or [11], p. 125, Theorem 11.30).

The topic of subsets of a (locally) compact group which are not measurable with respect to Haar measure is a wide field. Hewitt and Ross provide an instructive and far-reaching discussion of this topic in [11], pp. 226ff. The present question differs insofar as in this paper we are looking for the existence of nonmeasurable subgroups rather than just nonmeasurable subsets.

**Question 1.** [25] *Does every infinite compact group contain a nonmeasurable subgroup?*

For abelian compact groups COMFORT et al [3] showed the existence of nonmeasurable subgroups. See also KHARAZISHVILI [16]. For some partial answers in the noncommutative case see Gelbaum [6], 4.45.

## 1. The standard background material

We now present a systematic approach towards answering Question 1. First, we introduce some pertinent notation.

The terms  $|G|$  and  $\text{card}(G)$  equivalently denote the cardinality of  $G$ , notably when  $G$  is a group. Secondly if  $H$  is a subgroup of  $G$  and  $G/H$  is the set of cosets  $gH$ ,  $g \in G$ , then  $(G : H) \stackrel{\text{def}}{=} \text{card}(G/H)$  denotes the *index of  $H$  in  $G$* . We say that  $H$  has *countably infinite index (in  $G$ )* iff  $(G : H) = \aleph_0$ , and that  $H$  has *countable index (in  $G$ )* iff  $\text{card}(G/H) \leq \aleph_0$ .

**Proposition 1.1.** *Let  $G$  be a compact group and  $H$  a subgroup.*

(a) *If  $H$  has countably infinite index, then  $H$  is nonmeasurable. In particular,  $H$  is not a Borel subset.*

(b) *If  $H$  is measurable, then either it has measure 0, or it has positive measure in which case it is open (thus having finite index).*

(c) *If  $H$  has finite index in  $G$  and is not closed, then  $H$  is nonmeasurable. In particular, a countable index subgroup  $H$  of  $G$  is either closed with finite index or is nonmeasurable.*

(d) *If  $H$  is nonmeasurable in  $G$ , then  $\overline{H}$  is an open (and therefore finite index) subgroup of  $G$ .*

(e) *If  $H$  is a finite index subgroup,  $G = H \cup g_1H \cup \cdots \cup g_nH$ , then the largest normal subgroup  $N = H \cap g_1Hg_1^{-1} \cap \cdots \cap g_nHg_n^{-1}$  has finite index in  $G$ .*

(f) *Assume that  $H$  is nonmeasurable and that  $N$  is the largest normal subgroup contained in the open subgroup  $\overline{H}$ . Then  $N$  is open and  $H \cap N$  is dense in  $N$  and nonmeasurable in  $N$ .*

(g) *Assume that  $f: G \rightarrow G_1$  is a surjective morphism of compact groups and that  $H_1 \subseteq G_1$  is a nonmeasurable subgroup of countable index. Then  $H \stackrel{\text{def}}{=} f^{-1}(H_1)$  is a nonmeasurable subgroup of  $G$ .*

**Proof.** (a) (See [11], p. 227 and [25], Remark on p. 373.) Let  $\{g_1 = 1, g_2, \dots\}$  be a system of representatives for  $G/H$ , that is

$$(1) \quad G = \bigcup_{n=1}^{\infty} g_nH, \quad \text{a disjoint union.}$$

Suppose that  $H$  is measurable. Then  $g_nH$  is measurable for all  $n$  and  $\mu(g_nH) = \mu(H)$  by the invariance of Haar measure. So (1) implies

$$(2) \quad 1 = \mu(G) = \sum_{n=1}^{\infty} \mu(g_nH) = \sup_{N=1,2,\dots} \sum_{n=1}^N \mu(g_nH) = \sup_{N=1,2,\dots} N \cdot \mu(H).$$

In particular,  $\{N \cdot \mu(H) : N = 1, 2, \dots\}$  is a bounded set of nonnegative numbers, and this implies  $\mu(H) = 0$ . Then  $N \cdot \mu(H) = 0$  for all  $N = 1, 2, \dots$  and so  $\sup_{N=1,2,\dots} N \cdot \mu(H) = 0$ . This contradicts (2) and therefore our supposition must be false. That is,  $H$  is nonmeasurable.

(b) If  $H$  is measurable and has positive measure, then by [11], 20.17, p. 296, the group  $H = HH$  has inner points, and thus is open.

(c) If  $H$  has finite index in  $G$ , as in (a) above, let  $\{g_1 = 1, g_2, \dots, g_N\}$  be a system of representatives for  $G/H$ . Assume that  $H$  is measurable. Then  $1 = \mu(G) = \sum_{n=1}^N \mu(g_n H) = N \cdot \mu(H)$ . Thus  $\mu(H) = \frac{1}{N} > 0$ . Then  $H$  is an open subgroup by (b) and thus is also closed.

(d) Indeed,  $H$  is nowhere dense iff  $\overline{H}$  has no inner points. So  $\mu(H) = 0$ . Then  $H$  is a subset of a null set and therefore is measurable (and is a null set).

(e) This is straightforward.

(f)  $N$  is open and of finite index by (d) and (e). Now  $N = \overline{N \cap H}$  since  $N$  is open and  $H$  is dense in  $\overline{H}$ . Also, since  $N$  is an identity neighborhood,  $\overline{H} \subseteq HN \subseteq \overline{H}$ , that is,  $\overline{H} = HN = NH$ . There are elements  $h_1, \dots, h_k \in H$  such that  $\overline{H} = N \cup h_1 N \cup \dots \cup h_k N$  is a coset decomposition of  $\overline{H}$ .

Now suppose that  $N \cap H$  is measurable. Then  $h_k(N \cap H) = h_k N \cap H$  is measurable for all  $k = 1, \dots, m$ , and so  $(N \cap H) \cup (h_1 N \cap H) \cup \dots \cup (h_k N \cap H) = \overline{H} \cap H = H$  is measurable in contradiction to the hypothesis on  $H$ . If  $\mu$  is Haar measure on  $G$ , then  $(G : N)^{-1} \mu|_N$  is normalized Haar measure on  $N$ .

(g) If the index of  $H_1$  in  $G_1$  is infinite, then  $|G/H|$  is infinite, whence  $H$  is nonmeasurable by (a) above. If  $(G_1 : H_1) < \infty$ , then  $(G : H) < \infty$  since  $G/H \cong G_1/H_1$ . If  $H$  were measurable, then it would be open in  $G$  by (b) and thus  $H_1$  would be open in  $G_1$  which is not the case. So  $H$  is nonmeasurable.  $\square$

Regarding condition (c) above we should note right away that an infinite algebraically simple compact group such as  $\text{SO}(3)$  (see [14], Theorem 9.90) does not contain any proper finite index subgroups in view of (e) while, as we shall argue in Theorem 2.3 below, it does contain countably infinite index subgroups. On the other hand, a power  $A_5^{\mathbb{N}}$  with the alternating group  $A_5$  of 60 elements does not contain any countably infinite index subgroup as THOMAS shows in Theorem 1.10 of [28], while it does contain nonclosed proper finite index subgroups. In [17], KLEPPNER shows (in terms of homomorphisms onto finite discrete groups) that nonopen finite index normal subgroups are nonmeasurable.

In order to better understand the focus of our observations let us say that we may distinguish the following classes of compact groups:

Class 1: compact groups having subgroups of countably infinite index;

Class 2: compact groups having nonclosed subgroups of finite index;

Class 3: compact groups having in which every countable index subgroup is open closed.

In the direction of answering Question 1, the listing of known facts in Proposition 1.1 allows us to say that all groups in Classes 1 and 2 have nonmeasurable subgroups. The group  $\mathrm{SO}(3)$  is a member of Class 1 but is not in Class 2. The group  $A_5^{\mathbb{N}}$  belongs to Class 2 and not to Class 1. If  $G_1$  is a member of Class 1 and  $G_2$  is a member of Class 2, then  $G_1 \times G_2$  is a member of the intersection of the two classes. In the end we have to focus on Class 3, the complement of the union of the first two classes; however, it will serve a useful purpose to understand how big this union is and where familiar categories of compact groups are classified in this system.

The following discussion of examples show how members of Class 1 and 2 may arise. For this purpose let  $K$  be an arbitrary compact nonsingleton group. Let  $X$  be an infinite set endowed with the discrete topology, for instance  $X = \mathbb{N}$ . Then the compact group  $G = K^X$  has an alternative description. Indeed we consider  $X$  as a subset of its Stone Čech compactification  $\beta X$  and note that, due to the compactness of  $K$ , every element  $f \in G$ , that is, every function  $f: X \rightarrow K$  has a unique extension to a continuous function  $\bar{f}: \beta X \rightarrow K$ . The function  $f \mapsto \bar{f}: G \rightarrow C(\beta X, K)$  is an isomorphism of groups if we give  $C(\beta X, K)$  the pointwise group operations. If we endow  $C(\beta X, K)$  with the topology of pointwise convergence on the points of  $X$ , then  $C(\beta X, K)$  is a compact group and  $f \mapsto \bar{f}$  is an isomorphism of compact groups with the inverse  $F \mapsto F|_X$ . We shall identify  $G$  and  $C(\beta X, K)$  and note that  $G$  has a much finer topology, namely, that of uniform convergence on compact subsets of  $\beta X$  (to compact-open topology) giving us a topological group  $\Gamma$  with the same underlying group as  $G$ . Now let  $y \in \beta X$  and let  $H$  denote a proper closed subgroup of  $K$ . Then  $G_{y,H} = \{f \in \Gamma : f(y) \in H\}$  is a closed subgroup of  $\Gamma$ . We record the following lemma:

**Lemma 1.2.** *The following statements are equivalent:*

- (1)  $G_{y,H}$  is closed in  $G$ .
- (2)  $y \in X$ .

**Proof.** Since  $y \in X$  implies that, for the continuous projection  $p_y: G \rightarrow K$ ,  $p_y(f) = f(y)$ , the set  $G_{y,H}$  is just  $p_y^{-1}(H)$ , (1) follows trivially from (2).

Now suppose (1) is true and (2) is false. We must derive a contradiction. Since  $\beta X$  is zero dimensional (in fact extremally disconnected), the point  $y$  has a basis

$\mathfrak{U}$  of open-closed neighborhoods Let  $g \in K \setminus H$  and define, for each  $U \in \mathfrak{U}$ , a continuous function  $f_U: \beta X \rightarrow K$  by

$$(*) \quad f_U(z) = \begin{cases} g & \text{if } z \in \beta X \setminus U, \\ 1 & \text{if } z \in U. \end{cases}$$

Since  $G$  is compact, there is a cofinal net  $(U_j)_{j \in J}$  in  $\mathfrak{U}$  such that  $f = \lim_{j \in J} f_{U_j}$  exists in  $G$ . By  $(*)$  we have  $f \in G_{h,H}$ . Since  $y$  is not isolated in  $\beta X$  as (2) fails, also from  $(*)$  we have a net  $x_j \in X \setminus U_j$  converging to  $y$  such that  $f_{U_j}(x_j) = g$ . Let  $N$  be an open neighborhood of  $g$  in  $K$  with  $1 \notin N$ . The continuity of  $f$  implies the existence of a closed neighborhood  $W$  of  $y$  in  $\beta X$  such that  $f(W) \cap N = \emptyset$ . Now let  $k \in J$  be such that  $j \geq k$  implies  $x_j \in W$ . Then for  $i \geq k$  we have  $g = \lim_{j \geq k} f_{U_j}(x_i) = f(x_i) \in f(W)$ , and thus  $g \notin N$ , a contradiction.  $\square$

The argument shows in fact that for  $y \in \beta X \setminus X$ , the proper subgroup  $G_{y,H}$  is dense in  $G$ .

**Corollary 1.3.** *Every compact group  $G$  of the form  $G = K^X$ , for an infinite set  $X$  and a profinite group  $K$ , has nonmeasurable subgroups.*

**Proof.** Let  $H$  be a proper subgroup of  $K$  of finite index. Then for each  $y \in \beta X \setminus X$ , the subgroup  $G_{y,H}$  fails to be closed by Lemma 1.2. On the other hand, since  $G/G_{y,H} \cong K/H$  algebraically,  $G_{y,H}$  has finite index and thus is not measurable by 1.1(c).  $\square$

In particular,

*for each finite group  $F$  and each infinite set  $X$ , the profinite group  $F^X$  has non-measurable subgroups.*

As a an extension of Corollary 1.3 we mention the following observation which is obtained as a simple application of Proposition 1.1(g).

**Corollary 1.4.** *If  $G = \prod_{j \in J} G_j$  for a family of compact groups  $G_j$  and there is an infinite subset  $I \subseteq J$  such that  $G_j \cong K$  for all  $j \in I$  and for a profinite group  $K$ , then  $G$  has nonclosed finite index and therefore nonmeasurable subgroups.*

**Proof.** We may identify each  $G_j$  with  $K$  for  $j \in I$  and define the morphism  $f: G \rightarrow K^I$  as the obvious partial product. Then  $K^I$  has a nonmeasurable subgroup  $H_1$  by Corollary 1.3. So  $H \stackrel{\text{def}}{=} f^{-1}(H_1)$  is a nonmeasurable subgroup of  $G$  by Proposition 1.1(g).  $\square$

## 2. The case of countably infinite index subgroups

In [10], Corollary 1.2 we noted that every uncountable abelian group has a proper subgroup  $H$  of countable index. (See also [11], p. 227.) Accordingly, by Proposition 1.1(a), we have

**Proposition 2.1.** [3] *An infinite compact abelian group has a nonmeasurable subgroup.*  $\square$

A bit more generally, we have the following observation as an immediate consequence of Proposition 2.1 and Proposition 1.1(c).

**Corollary 2.2.** *If the algebraic commutator group  $G'$  of a compact group  $G$  has infinite index in  $G$ , then  $G$  has nonmeasurable subgroups. If the subgroup  $G'$  has finite index it is either open closed or nonmeasurable.*  $\square$

Here is a partial answer to Question 1:

**Theorem 2.3.** *Every infinite compact group  $G$  that is not profinite has a subgroup of countable index and thus contains a nonmeasurable subgroup.*

**Proof.** Assume that  $G$  is not profinite. Let  $N$  be a closed normal subgroup of  $G$  such that  $C = G/N$  is an infinite compact Lie group ([14], Corollary 2.43).

We will show that  $C$  contains a subgroup with countable index; for the pullback to  $G$  of a subgroup of countable index in  $C$  yields a subgroup of countable index in  $G$ . A subgroup of the identity component  $C_0$  of  $C$  with countable index has countable index in  $C$  as  $C/C_0$  is finite. Now  $C_0$  is a compact connected Lie group and we claim that it has a subgroup of countable index.

The commutator subgroup  $C'_0$  of  $C_0$  is closed ([14], Theorem 6.11) and so, if  $C'_0 \neq C_0$ , then  $C_0/C'_0$  is a connected abelian Lie group and thus is infinite and therefore contains a subgroup of countable index by Lemma 1.2. Thus  $C_0$  has a countable index subgroup.

Next we assume that  $C'_0 = C_0$  and thus that  $C_0$  is semisimple and there is a homomorphism onto a compact connected simple and centerfree Lie group  $S$ . (See [14], Theorem 6.18.) Now  $S$  has no subgroup of finite index, because if  $H$  were a finite index subgroup of  $S$ , then the intersection of the finitely many conjugates of  $H$  would be a finite index normal subgroup which cannot exist (see [14], Theorem 9.90).

Now  $S$  has a faithful linear representation as an orthogonal matrix group (cf. [14], Corollary 2.40). By a theorem of KALLMAN [15], therefore  $S$  has a faithful algebraic representation as a permutation group on  $\mathbb{N}$ . Since all orbits of  $S$  on

$\mathbb{N}$  are countable or finite, the isotropy groups all have countable or finite index. Since  $S$  has no finite index subgroups, it does have a countable index subgroup which finally pulls back to a finite index subgroup of  $G$ .  $\square$

At this stage there remains the case of profinite groups.

Before we address this case let us observe that even in the compact abelian case the issue of countably infinite index subgroups is far from trivial. From [12] we quote (cf. also [14], Theorem 8.99):

**Theorem 2.4.** *There is a model of set theory in which there is a compact group  $G$  with weight  $\aleph_1 = 2^{\aleph_0}$  such that the arc component factor group  $\pi_0(G) = G/G_a$  is algebraically isomorphic to  $\mathbb{Q}$ . In particular, the arc component of  $G_a$  of the identity is a countably infinite index subgroup and, accordingly, is nonmeasurable.*

### 3. Profinite groups

We record next that not all compact groups have countably infinite index subgroups:

**Example 3.1.** Let  $A_5$  the alternating group on five elements, the smallest finite simple nonabelian group. Then  $G = (A_5)^{\mathbb{N}}$  has no subgroups of countably infinite index.

This follows from Theorem 1.10 of Thomas [28]. In fact Thomas classifies infinite products of finite groups in which every subgroup of index  $< 2^{\aleph_0}$  is necessarily open; such groups do not have countably infinite index subgroups. From Corollary 1.3 it follows that  $G$  in Example 3.1 has nonmeasurable subgroups.

The literature provides some guidance on the situation of finite index subgroups. We begin with a result of M. G. SMITH and J. S. WILSON [27].

**Proposition 3.2.** *Let  $G$  be a profinite group. Then all finite index normal subgroups are open if and only if there are countably many finite index subgroups.*  $\square$

Since the cardinality of the set open normal subgroups in a profinite group is the weight of the group, an immediate corollary is

**Corollary 3.3.** *Let  $G$  be a profinite group of uncountable weight. Then  $G$  contains nonclosed finite index subgroups and these are, accordingly, nonmeasurable.*  $\square$

In fact as a consequence of PETERSON's Theorem 1.2(2) of [20] it has been known for some time that these large profinite groups contain at least  $2^{2^{\aleph_0}}$  nonmeasurable subgroups.

The focus therefore is on profinite groups with countably many finite index normal subgroups, and in accordance with some authors we use the following definition (see [21], Section 4.2, pp. 124ff.)

**Definition 3.4.** A *strongly complete group* is a profinite group in which every finite index subgroup is open.

We summarize our findings:

**Theorem 3.5.** A compact group in which every subgroup is measurable is a strongly complete group.

**Proof.** Let  $G$  be a compact group. If it is not totally disconnected, then it has a subgroup of countably infinite index by Theorem 2.3 and thus a nonmeasurable subgroup. If all subgroups of  $G$  are measurable, then  $G$  is profinite. If it has a subgroup of finite index that fails to be open closed, then such a subgroup is nonmeasurable by Proposition 1.1(c). Thus all finite index subgroups of  $G$  are open closed and so  $G$  is a strongly complete group.  $\square$

So Question 1 reduces to

**Question 2.** Does every infinite strongly complete group contain a nonmeasurable subgroup?

We keep in mind that Smith and Wilson [27] showed that a profinite group is strongly complete if and only if it has only countably many finite index subgroups. Such a group is necessarily metric. Segal and Nikolov [19] showed that all topologically finitely generated metric profinite groups are strongly complete, as had been conjectured by Serre.

Typical examples in this class of groups are countable products of pairwise nonisomorphic simple finite groups. A result of SAXL's and WILSON's [26] says:

**Proposition 3.6.** Let  $\{G_n : n \in \mathbb{N}\}$  be a sequence of finite simple nonabelian groups and  $G = \prod_{n \in \mathbb{N}} G_n$ . Then the following conditions are equivalent:

- (i) Infinitely many of the  $G_n$  are isomorphic.
- (ii)  $G$  is not strongly complete.  $\square$

We have observed in Theorem 3.5 that (ii) implies



(iii)  $G$  has nonmeasurable subgroups.

The implication “(i) implies (ii)” also follows from our Corollary 1.4 above.

#### 4. Metric compact groups

We propose to calculate the cardinality of the set  $\mathcal{S}(G)$  of (not necessarily closed!) subgroups of  $G$ . We let  $\mathfrak{c} = 2^{\aleph_0}$  denote the cardinality of the continuum and  $\mathcal{B}(G)$  the set of all Borel subsets of  $G$ .

**Proposition 4.1.** If  $X$  is an infinite 2nd countable metric space, then

$$\text{card}(\mathcal{B}(X)) \leq \mathfrak{c}.$$

**Proof.** In [1], Exercise 4 c), §6, Chap. 9 it is established that the cardinality of the set of Borel subsets of a metric second countable space is  $\leq \mathfrak{c}$ .  $\square$

**Remark 4.2.** For Haar measure  $\mu$  on a compact metric group  $G$ , a subset  $X$  is measurable iff there are sets  $B_1, B_2 \in \mathcal{B}(G)$  such that  $B_1 \subseteq X \subseteq B_2$  such that  $\mu(B_2 \setminus X) = 0 = \mu(X \setminus B_1)$ .  $\square$

(See e.g. [24], 10.10; the argument given there is quite general.)

Since the cardinality of an infinite compact metric group is  $\mathfrak{c}$  and therefore the cardinality of the set of its subsets is  $2^{\mathfrak{c}}$ , the following observation is trivial.

**Lemma 4.3.** If  $G$  is a compact metric group,  $G/N$  an infinite quotient group, and  $H$  any subgroup of  $G$  then  $\max\{\text{card}(\mathcal{S}(H)), \text{card}(\mathcal{S}(G/N))\} \leq \text{card}(\mathcal{S}(G)) \leq 2^{\mathfrak{c}}$ .  $\square$

The following is easily established.

**Lemma 4.4.** A vector space of the infinite dimension  $\aleph$  has  $2^{\aleph}$  vector subspaces.

We now prove that an infinite compact metric group has as many subgroups as it has subsets.

**Theorem 4.5.** Let  $G$  be an infinite metric compact group. Then

$$\text{card}(\mathcal{S}(G)) = 2^{\mathfrak{c}}.$$

**Proof.** (i) By Zelmanov’s Theorem [30],  $G$  contains an infinite abelian subgroup  $A$  which we may assume to be closed. Then  $A$  is a compact metric abelian group. If the assertion of the Theorem is true for abelian groups, then it is true in general by Lemma 4.3 Thus we assume from here on that  $G$  is abelian.

(ii) By Lemma 4.3, we have

$$\max\{\text{card}(\mathcal{S}(G_0)), \text{card}(\mathcal{S}(G/G_0))\} \leq \text{card}(\mathcal{S}(G)).$$

If  $G_0 \neq \{0\}$ , by Lemma 4.3, we may assume  $G$  to be connected. Then  $G$  is divisible and the torsion free summand is a  $\mathbb{Q}$ -vector space of dimension  $\mathfrak{c}$ . Hence  $G$  has at least  $2^\mathfrak{c}$  subgroups. So from here on we may and will assume that  $G$  is profinite,

(iii) For any prime  $p$  Let  $G_p$  be the  $p$  primary component of the abelian profinite group  $G$ . Then  $G \cong \prod_{p \text{ prime}} G_p$  [14], Corollary 8.8(iii). There are two cases:

(A) There is a  $p$  such that  $G_p$  is infinite, or

(B) for all  $p$  the group  $G_p$  is finite.

(iv) In Case (A) we invoke Lemma 3.4 and assume that  $G$  is a pro- $p$ -group. The  $p$ -socle of  $\widehat{G}$  is a  $\text{GF}(p)$ -vector space of dimension  $p$ -rank  $G$  (see [14] A1,21).

Then again we have two cases: (a) The  $p$ -rank, that is the  $\text{GF}(p)$ -dimension of the  $p$ -socle of  $\widehat{G}$  is finite, or else, (b) it is infinite.

In case (a)  $\widehat{G}$  has a direct summand isomorphic to  $\mathbb{Z}_p$  and by Lemma 3.4 again, we may assume that  $G = \mathbb{Z}_p$ . In this case  $G$  has a subgroup which is isomorphic to a  $\mathbb{Q}$ -vector space of dimension  $\mathfrak{c}$  and thus by Lemma 4.4,  $G$  has at least  $2^\mathfrak{c}$  subgroups.

In case (b), by duality, there is a surjective morphism  $G \rightarrow \mathbb{Z}(p)^\mathbb{N}$ . Then by Lemma 3.4 we assume that  $G = \mathbb{Z}(p)^\mathbb{N}$ . Then  $G$  is algebraically a  $\text{GF}(p)$ -vector space of dimension  $\mathfrak{c}$ . By Lemma 4.4 again,  $G$  has at least  $2^\mathfrak{c}$  subgroups.

(v) We are now reduced to case (B) and so, since  $G$  is infinite, there are infinitely many primes  $p_1, p_2, \dots$  such that  $G_{p_n} \neq \{0\}$ ,  $G_{p_n}$  finite and we may assume w.l.o.g.  $G_{p_n} = \mathbb{Z}(p_n)$ . So we have to prove that for  $G = \prod_{n=1}^\infty \mathbb{Z}(p_n)$  we have  $\text{card}(\mathcal{S}(G)) = 2^\mathfrak{c}$ . Now  $\bigoplus_{n=1}^\infty \mathbb{Z}(p_n)$  is the torsion group  $T$  of  $G$ , and we claim that the torsion free rank of  $G/T$  is  $2^\mathfrak{c}$ . This claim is equivalent to saying that  $\text{card } G/T = 2^\mathfrak{c}$ . But since  $\text{card } G = 2^\mathfrak{c}$  and  $T$  is countable, this is clear.  $\square$

**Corollary 4.6.** *Every infinite compact group has a subgroup which is not a Borel subset.*

**Proof.** By Proposition 4.1 and Theorem 4.5, every infinite compact *metric group* has more subgroups than it has Borel subgroups.

By the results preceding Section 4, every compact group which fails to be a metric profinite group has a subgroup which is nonmeasurable for Haar measure. Since all Borel subgroups are Haar measurable, none of these is a Borel subgroup.  $\square$

## References

- [1] Bourbaki, N., «Topologie générale», Chap 9, Hermann, Paris, 1958.
- [2] Bourbaki, N., «Intégration», Chap. 1,2,3 et 4, Hermann, Paris, 2me éd, 1965.
- [3] Comfort, W. W., S. U. Raczkowski, and F. J. Trigos-Arrieta, *Making group topologies with, and without, convergent sequences*, Appl. Gen. Topol. **7** (2006), 109–124.
- [4] Ferrer, Maria V., Hernández, Salvador, *Dual topologies on non-abelian groups*, Topology and its Applications, Special Issue in Honor of Dikran Dikranjan (2012), to appear.
- [5] Gartside, P., and M. Smith, *Counting the closed subgroups of profinite groups*, arXiv:0809.4734v1 [math.GR] 27 Sep 2008
- [6] Gelbaum, B. R., “Problems in real and complex analysis,” Problem Books in Analysis, Springer-Verlag, New York etc., 1992, x+488p.
- [7] Gierz, G., K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, “Continuous Lattices and Domains,” Encycl. of Math. and its Appl. **93**, Cambridge Univ. Press, 2003, xxxvi+591 pp.
- [8] Hart, J. E., and K. Kunen, *Compactifications of Non-Abelian Groups*, Topology Proceedings 26 (2001-2002), 593–626
- [9] Hernández, S., *Questions raised at the conference “Algebra meets Topology,”* Barcelona, July 2010. [hernande@mat.uji.es](mailto:hernande@mat.uji.es)
- [10] Hernández, S., K. H. Hofmann, and S. A. Morris, *The weights of closed subgroups of a locally compact group*, J. of Group Theory **15** (2014), 613–630.
- [11] Hewitt, E., and K. A. Ross, “Abstract Harmonic Analysis I,” Grundlehren **115** Springer Verlag, Berlin etc., 1963.
- [12] Hofmann, K. H., Arc components of locally compact groups are Borel sets, Bull. Austr. Math. Soc. **65** (2002), 1–8
- [13] Hofmann, K. H., and S. A. Morris, *A structure theorem on compact groups*, Math. Proc. Camb. Phil. Soc. **130** (2001), 409–426.
- [14] —, “The Structure of Compact Groups,” de Gruyter Studies in Math. **25**, Berlin, 3<sup>rd</sup> Edition 2013, xxii+924 pp.
- [15] Kallman, R. R., *Every reasonably sized matrix group is a subgroup of  $S_\infty$* , Fund. Math. 164 (2000), 35–40.
- [16] Kharazishvili, A. B., *On thick subgroups of uncountable  $\sigma$ -compact locally compact commutative groups*, Topology and its Applications **156** (2009), 2364–2369.
- [17] Kleppner, A., *Measurable homomorphisms of locally compact groups*, Proc. Amer. Math. Soc. **106** (1989), 391–395.
- [18] Montgomery, D. and L. Zippin, “Topological Transformation Groups,” Interscience, New York, 1955.

- [19] Nikolov, N., and D. Segal, *On finitely, generated profinite groups, I: Strong completeness and uniform bounds* Ann. of Math. **165** (2007), 171–238.
- [20] Peterson, H. L., *Extensions of Haar measure to relatively large nonmeasurable subgroups*, Trans. Amer. Math. Soc. **228** (1977), 359–370.
- [21] Ribes, L., and P. Zaleskii, “Profinite Groups,” Springer-Verlag, Berlin etc., 2000, xiv+435 pp. 2nd edition 2010, xvi+464 pp.
- [23] Rudin, W., “Fourier Analysis on Groups,” Interscience, New York, 1962.
- [24] —, “Principles of Mathematical Analysis,” MacGraw- Hill, New York etc., 1953, 1964.
- [25] S. Saeki and Karl Stromberg, *Measurable subgroups and nonmeasurable characters*, Math.Scnd. **57** (1985), 359–374.
- [26] Saxl J., and J. S. Wilson, *A note on popers in simple groups* Math. Proc. Cambridge Phil. Soc. **122** (1997), 91–94
- [27] Smith, M. G., and J. S. Wilson *On subgroups of finite index in compact Hausdorff groups*, Archiv d. Math. **80** (2003), 123–129.
- [28] Thomas, S., *Infinite products of finite simple groups II*, J. of Group Theory **2** (1999) 401–434.
- [29] Zelenyuk, Yevhen, “Ultrafilters and Topologies on Groups,” DeGruyter Expositions in Math **50**, Berlin, 2011
- [30] Zelmanov, E. I., *On periodic compact groups*, Israel J. of Math. **77** (1992), 83–95.

#### AUTHORS’ ADRESSES

Salvador Hernández  
hernande@mat.uji.es

Universitat Jaume I, INIT and Depto de Matemáticas, Campus de Riu Sec, 12071 Castellón, Spain

Karl H. Hofmann, Corresponding author  
hofmann@mathematik.tu-darmstadt.de

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany.

Sidney A. Morris  
morris.sidney@gmail.com

School of Science, IT, and Engineering, Federation University Australia, Victoria 3353, Australia, and School of Engineering and Mathematical Sciences, La Trobe University, Bundoora, Victoria 3086, Australia